Lecture 2

Tuesday, 27 January 2026 8:43 AM

Theorem 1: Suppose $P = \{x \in \mathbb{R}^n : Ax \geq b\}$ is nonempty & has an extreme pt. Consider the LP: $\{\min c^T x : x \in P\}$. Then either the optimal value is $-\infty$, or there is an optimal extreme pt.

Corollary: If P is a non-empty polytope, there exists a optimal extreme pt.

(w/o proof)

But then we have an (inefficient) algo for solving LPs on polytopes:

$$\text{LP}: \quad \min c^T x$$
$$Ax \leq b, \quad \text{feasible region is a polytope}$$
$$A \in \mathbb{R}^{m \times n}$$

Then we know there is an optimal bfs.

We know how to enumerate all basic solutions

So:
- enumerate all basic solutions, since at a basic solution, n (linearly independent) constraints. So for each choice of n constraints (say $a_1, \dots, a_n$), check if l.i., if so, solve $a_1 x = b_1, \dots, a_n x = b_n$
- check if each is feasible
- pick basic feasible soln with minimum value.

Can be done in time $O(m^n)$.

Not a great algo, just brute force, enumerate all vertices of the polytope.

Instead, can we search the vertices in a more structured way? This is the Simplex algorithm.

Simplex Algorithm

Generally described for LPs in standard form.

$$\min c^T x$$
$$Ax = b, \quad A \in \mathbb{R}^{m \times n}$$
$$x \geq 0$$

Will assume:

1. rows of A are linearly independent (thus $m \leq n$)
2. polyhedron is non degenerate: at any point in P, at most n constraints are tight (hence, at most $m-n$ variables are zero, or at least m variables are non-zero)

So, at a bfs x: $Ax = b$

exactly m variables are non-zero.

These m variables are called 'basic' variables

Simplex method boils down to iterating over diff. choices of the m 'basic', non-zero variables, & finding the 'best' (which gives the least objective value).

Eg.

$$\begin{array}{l}
\overset{\text{basic}\ \downarrow\qquad\downarrow}{\boxed{\begin{array}{l} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{array}}}
\begin{array}{l} \overbrace{+\, a_{13}x_3 + a_{14}x_4}^{\text{non-basic, set to zero}} = b_1 \\ +\, a_{23}x_3 + a_{24}x_4 = b_2 \end{array} \\
x_1 \geq 0 \\
x_2 \geq 0 \\
x_3 \geq 0 \\
x_4 \geq 0
\end{array}$$

So if basic vars. $= x_1, x_2$, we are solving

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

for the resulting x to be a bfs, the matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ must have full rank, or equivalently, the m columns in A corresponding to the basic vars. must be linearly independent.

This is necessary & sufficient for basic solutions

Theorem: For polyhedron $P = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$ for $A \in \mathbb{R}^{m \times n}$ has rank m, $x \in \mathbb{R}^n$ is a basic solution iff $Ax = b$, & there exist indices $B(1), \dots, B(m)$ s.t.

(I) $x_i = 0 \quad \forall i \notin \{B(1), \dots, B(m)\}$

(II) $A_{B(1)}, \dots, A_{B(m)}$ are linearly independent.

(proof skipped)

Thus, for an LP in std. form where $A \in \mathbb{R}^{m \times n}$ has rank m, here is an alternate way of generating all basic solns:

- consider all choices of m linearly independent columns $A_{B(1)}, \dots, A_{B(m)}$ (corr. to choice of basic variables)
- solve $\begin{bmatrix} A_{B(1)} & \cdots & A_{B(m)} \end{bmatrix} \begin{bmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{bmatrix} = \begin{bmatrix} b_{B(1)} \\ \vdots \\ b_{B(m)} \end{bmatrix}$
- set $x_i = 0$ for $i \notin \{B(1), \dots, B(m)\}$

Note:
- if $x_i \geq 0 \ \forall i \in [n]$, then such a soln is a bfs
- this generates all bfs as well.

Some defns:
- the vars. $x_{B(1)}, \dots, x_{B(m)}$ chosen to be non-zero are called the basic variables
- the indices $B(1), \dots, B(m) \subseteq [n]$ are called the basis (since columns $A_{B(1)}, \dots, A_{B(m)}$ form a basis for $\mathbb{R}^m$)
- will use $B = \begin{bmatrix} A_{B(1)}, \dots, A_{B(m)} \end{bmatrix}$, $x_B = \begin{bmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{bmatrix}$
- so if B is a basis, the basic soln is given by $x_B = B^{-1} b$

Defn: Two basic solns. x & y are adjacent if $\exists$ $(n-1)$ l.i. constraints that are tight at both.

Assume we're at a bfs x w/ basic vars. $B(1), \dots, B(m)$

So for all $j \notin \{B(1), \dots, B(m)\}$, the constraint $x_j \geq 0$ is tight.

To go to the next bfs, we will "untighten" one constraint, i.e, choose a non-basic variable x_j, start increasing it, while keeping all other non-basic variables unchanged, i.e., from X, we move in a direction d s.t.:

$$d_j = 1,$$
$$d_k = 0 \quad \text{for } k \notin \{B(1), \dots, B(m)\},\ k \neq j$$

Thus we want $A(x + \lambda d) = b$ for $\lambda > 0$,

or $Ad = 0$

or $A_j + B d_B = 0$ or $d_B = -B^{-1} A_j$

Recall that by non-degeneracy, all basic variables are non-zero,

thus $x_k > 0 \quad \forall k \in \{B(1), \dots, B(m)\}$

hence for d defined by

$$d_j = 1, \quad j \text{ is a non-basic variable}$$
$$d_k = 0, \quad k \notin \{B(1), \dots, B(m)\},\ k \neq j.$$
$$d_B = -B^{-1} A_j,$$

& for small λ, $(x + \lambda d)$ remains feasible.

(Q. is $x - \lambda d$ feasible?)

How does the cost change?

rate of change in cost $= c^T d = c_j + c_B^T d_B = c_j - c_B^T B^{-1} A_j$

This is called the reduced cost of j, if B is the basis.

Defn: For a fixed basis B, define the reduced cost of a non-basic variable x_j as:

$$\bar{c}_j = c_j - c_B^T B^{-1} A_j$$

(HW: Show that if j is a basic variable, the reduced cost $\bar{c}_j = 0$)

Theorem: Let $P = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$ be a non-degenerate polyhedron, & cost fn $\min c^T x$. Let x be a bfs w/ basis $B(1), \dots, B(m)$, & $\bar{c}$ be the vector of reduced costs. Then:

$$\bar{c} \geq 0 \quad \text{iff} \quad x \text{ is optimal}$$

Proof: Suppose $\bar{c} \geq 0$. We will show x is optimal.

Fix $y \in P$, will show that $c^T y \geq c^T x$

Let $d = y - x$. Note that for $j \notin B$, $x_j = 0$, hence $d_j \geq 0$.

Will show that $c^T d = \sum_{j \notin B} \bar{c}_j d_j$

But we know that $\bar{c} \geq 0$, $d_j \geq 0$ for $j \notin B$. Hence $c^T d \geq 0$ hence $c^T y \geq c^T x$.

Now
$$\sum_{j \notin B} \bar{c}_j d_j = \sum_{j \notin B} (c_j - c_B^T B^{-1} A_j) d_j$$
$$= \sum_{j \notin B} c_j d_j - c_B^T B^{-1} \sum_{j \notin B} A_j d_j$$

$$Ad = 0 \Rightarrow \sum_{j \notin B} A_j d_j = -\sum_{j \in B} A_j d_j$$

hence
$$\sum_{j \notin B} \bar{c}_j d_j = \sum_{j \notin B} c_j d_j + c_B^T B^{-1} \sum_{j \in B} A_j d_j$$
$$= \sum_{j \notin B} c_j d_j + c_B^T B^{-1} B d_j$$
$$= c^T d$$

This proves that if $\bar{c} \geq 0$, x is an optimal bfs.

Now suppose $\exists$ non-basic variable x_j s.t. $\bar{c}_j < 0$. We want to show x is not optimal.

As just shown, $\sum_{j \notin B} \bar{c}_j d_j = c^T d$.

Then consider d:
$$d_j = 1 \quad (\text{where } \bar{c}_j < 0)$$
$$d_k = 0 \quad \text{for other non-basic vars}$$
$$d_B = -B^{-1} A_j$$

Then $c^T d = \bar{c}_j d_j < 0$

Hence for $\lambda > 0$, $c^T(x + \lambda d) < c^T x$.

HW: Can check that for small $\lambda > 0$, $c^T(x + \lambda d) \in P$. ▨

Hence, simplex algo:

1. Start w/ a bfs x, basis B (Q1: how do you find one?)
2. Compute reduced costs $\bar{c}_j \ \forall j \notin B$
   If $\bar{c}_j \geq 0 \ \forall j \notin B$, x is optimal
   Else $\exists j : \bar{c}_j < 0$
3. Pick $j : \bar{c}_j < 0$, Compute d: (Q2: which one?)
   $d_j = 1$, $d_k = 0 \ \forall k \neq j, k \notin B$, $d_B = -B^{-1} A_j$
   (index j enters the basis)
4. Compute $\lambda^* = \max\{\lambda : x + \lambda d \in P\}$
   (Let $l = \operatorname{argmin}_{k \in B,\, d_k < 0} -\frac{x_k}{d_k}$, then $\lambda^* = -\frac{x_l}{d_l}$)
5. New bfs $x' = x + \lambda^* d$
   basis $B' = B \cup \{j\} \setminus \{l\}$
   go back to step 2.

(Q3: Is B' a valid basis, & x' the corresponding bfs?)

Example:

$$\min \ 2x_1 - 3x_2 - x_3$$
$$\text{s.t.} \quad x_1 + x_2 + 2x_3 = 1$$
$$x_1, x_2, x_3 \geq 0$$

here $m = 1$, $n = 3$, so size of basis $= m = 1$

let's choose $B = \{1\}$ as our starting basis.

(1) so setting $x_2 = x_3 = 0$, we solve: $x_1 = 1$

this is a bfs.

reduced costs for nonbasic variables:

$$\bar{c}_j = c_j - c_B^T B^{-1} A_j, \quad \text{here } c_B = 2, B^{-1} = 1$$

so
$$\bar{c}_2 = -3 - 2.1.1 = -5$$
$$\bar{c}_3 = -1 - 2.1.2 = -5$$

so $(1, 0, 0)$ is not optimal.

let's bring x_3 into basis

then direction $d = (-2, 0, 1)$

(where $d_1 = -B^{-1} A_j = -2$)

& $\lambda^* = -\min_{k \in B,\, d_k < 0} \frac{x_k}{d_k} = -\frac{1}{-2} = \frac{1}{2}$

so new bfs is $(x + \lambda^* d) = (1, 0, 0) + \frac{1}{2}(-2, 0, 1)$

$= (0, 0, \frac{1}{2})$

& basis is x_3.

so is $(0, 0, \frac{1}{2})$ an optimal solution?

let's compute reduced costs for non-basic vars.

$$\bar{c}_j = c_j - c_B^T B^{-1} A_j, \quad c_B = -1, B^{-1} = \tfrac{1}{2}$$

so
$$\bar{c}_1 = 2 - (-1 . \tfrac{1}{2} . 2) = 3 > 0$$
$$\bar{c}_2 = -3 - (-1 . \tfrac{1}{2} . 1) = -3 + \tfrac{1}{2} = -2.5 < 0$$

so now we bring x_2 into basis

$$d = (0, 1, -\tfrac{1}{2})$$
$$d_3 = -B^{-1} A_j = -\tfrac{1}{2}$$
$$\lambda^* = -\min_{k \in B,\, d_k < 0} \frac{x_k}{d_k} = 1$$

so new bfs is $(0, 0, \frac{1}{2}) + 1.(0, 1, -\frac{1}{2}) = (0, 1, 0)$

is this optimal?

$$\bar{c}_j = c_j - c_B^T B^{-1} A_j, \quad c_B = -3, B^{-1} = 1$$
$$\bar{c}_1 = 2 - (-3 . 1 . 1) = 5$$
$$\bar{c}_3 = -1 - (-3 . 1 . 2) = 5$$

since $\bar{c} \geq 0$, the bfs $(0, 1, 0)$ is optimal.